

# Existence of minimizers of functionals involving the fractional gradient in the absence of compactness, symmetry and monotonicity

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## Abstract

We establish general assumptions under which a constrained variational problem involving the fractional gradient and a local nonlinearity admits minimizers.

## 1 Introduction

For a prescribed number  $c > 0$  and  $0 < s < 1$ , we consider the following constrained minimization problem :

$$\inf\{J(u) : u \in S_c\} = I_c \quad (1.1)$$

$$\begin{aligned} J(u) &= \frac{1}{2} \int |\nabla_s u|^2 - \int F(x, u), \\ |\nabla_s u|_2^2 &= \int |\nabla_s u|^2 = C_{N,s} \int \int \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \end{aligned}$$

$F$  is a carathéodory function, and

$$S_c = \{u \in H^s(\mathbb{R}^N) : \int u^2 = c^2\}.$$

Under some additional regularity assumptions on  $F$ , solutions of (1.1) satisfy the following fractional elliptic equation :

$$\Delta^s u + f(x, u) + \lambda u = 0 \quad (1.2)$$

where  $F(x, t) = \int_0^t f(x, p) dp$  and  $\lambda$  is a Lagrange multiplier. Solutions of (1.1) can also be viewed as standing waves of the following nonlinear fractional Schrödinger equation

$$\begin{cases} i\partial_t \Phi(t, x) + f(x, |\Phi|) + \Delta_{xx}^s \Phi = 0 \\ \Phi(0, x) = \Phi^0(x). \end{cases} \quad (1.3)$$

Despite the importance of (1.2) and (1.3) in many domains, there are only results the particular cases :  $N = 1, s = \frac{1}{2}$  and  $f(x, s) = s^\alpha, [1, 2]$ . Let us point out that when  $N = 3, s = \frac{1}{4}$ , (1.3) models water waves, semilunar heart valve vibrations and neural systems. When  $s = \frac{3}{4}$ , it governs water waves with surface tension, [5]. More generally, equations (1.2) and (1.3) arise in numerous models from mathematical physics, mathematical biology, finance, inhomogenous porous material, geology, hydrology, dynamics of earthquakes, bioengineering, chemical engineering, neural networks and medicine, [5, 6] and references therein.

In this paper, we address the question of existence of minimizers of (1.1) in the absence of compactness, symmetry and monotonicity. This considerably extends the main result obtained by the author in [5], where the integrand  $F$  has a nice combination of monotonicity and symmetry properties, which enabled us to obtain the compactness of Schwarz minimizing sequences. In the present work, we will prove the above property for any minimizing sequence of (1.1) without requiring any symmetry or monotonicity properties of the integrand.

Our main result is :

**Theorem 1.1.** Suppose that the function  $F : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function verifying :

(F0)  $\forall x \in \mathbb{R}^N, t \in \mathbb{R}, \exists A, A' > 0$  and  $0 < \ell < \frac{4s}{N}$  such that :

$$0 \leq F(x, t) \leq A(t^2 + |t|^{\ell+2})$$

and

$$0 \leq \partial_2 F(x, t) \leq A'(|t| + |t|^{\ell+1})$$

(F1)  $\exists \Delta > 0, S > 0, R > 0, \alpha > 0$   $p \in [0, 2)$  such that :

$$F(x, t) > \Delta |x|^{-p} |t|^\alpha \text{ for } |x| \geq R, |t| < S,$$

where

$$N + 2s > \frac{N}{2}\alpha + p,$$

$$(F2) \ F(x, \theta t) \geq \theta^2 F(x, t) \ \forall x \in \mathbb{R}^N, t \in \mathbb{R} \ \theta \geq 1.$$

There exists a periodic function  $F^\infty(x, t)$  (i.e  $\exists z \in \mathbb{Z}^N$  such that  $F^\infty(x + z, t) = F^\infty(x, t) \ \forall x \in \mathbb{R}^N, t \in \mathbb{R}$ ) satisfying (F1) such that :

$$(F3) \ \text{There exists } 0 < \beta < \frac{4s}{N} \text{ such that } \lim_{|x| \rightarrow \infty} \frac{F(x, t) - F^\infty(x, t)}{t^2 + |t|^{\beta+2}} = 0$$

uniformly for any  $t$ .

$$(F4) \ \text{There exists } B, B' \text{ and } 0 < \gamma < \ell < \frac{4s}{N} \text{ such that}$$

$$0 \leq F^\infty(x, t) \leq B(|t|^{\gamma+2} + |t|^{\ell+2})$$

and

$$0 \leq \partial_2 F^\infty(x, t) \leq B'(|t|^{\gamma+1} + |t|^{\ell+1})$$

$$\forall x \in \mathbb{R}^N, t \in \mathbb{R}.$$

$$(F5) \ \text{There exists } \sigma \in (0, \frac{4s}{N}) \text{ such that}$$

$$F^\infty(x, \theta t) \geq \theta^{\sigma+2} F^\infty(x, t)$$

$$\forall \theta \geq 1, x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

(F6)  $F^\infty(x, t) \leq F(x, t) \ \forall x \in \mathbb{R}^N, t \in \mathbb{R}$ , with strict inequality in a measurable set having a positive Lebesgue measure.

Then there exists  $u_c \in S_c$  such that

$$J(u_c) = I_c.$$

**Theorem 1.2** If (F1) holds true for  $F^\infty$ , (F4) and (F5) are satisfied, then there exists  $u_c \in S_c$  such that

$$J^\infty(u_c) = I_c^\infty, \text{ where } J^\infty(u) = \frac{1}{2} \int |\nabla_s u|^2 - \int F^\infty(x, u)$$

and

$$I_c^\infty = \inf\{J^\infty(u) : u \in S_c\}. \quad (1.4)$$

Our proofs of the above results are based on a variant of the breakthrough concentration-compactness principle (appendix).

Our line of attack consists of the following steps :

In order to prove that vanishing cannot occur, it is sufficient to show the strict negativity of the value of the infimum (Lemma 3.2). Then, to rule

out dichotomy, we will first prove that the minimization problem (1.4) is achieved

(S1)

and that :

$$I_c < I_c^\infty \quad \forall c > 0 \quad (S2)$$

$$I_c \leq I_{c-a} + I_a \quad \forall a \in (0, c) \quad (S3)$$

(S2) and (S3) imply the strict subadditivity inequality

$$I_c < I_{c-a}^\infty + I_a \quad \forall a \in (0, c) \quad (S4)$$

On the other hand, we will prove that thanks to our assumptions on  $F$ , we certainly have for any minimizing sequence  $(u_n)$  of (1.1) that :

$$J(u_n) \geq J(u_{n,1}) + J^\infty(u_{n,2}) - g(\delta) \quad (S5)$$

where  $g(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

The latter requires of course a deep and subtle study of the functionals  $J$  and  $J^\infty$  (Lemma 3.1).

Finally the continuity of  $I_c$  and  $I_c^\infty$  enables us to deduce that (S5) implies the following inequality :

$$I_c \geq I_a + I_{c-a}^\infty. \quad (S6)$$

(S4) together with (S6) yield to a contradiction.

Once one knows that compactness is the only plausible alternative, the strict inequality (S2) will be very helpful to conclude that any minimizing sequence of (1.1) is compact (up to a subsequence). These issues were heuristically discussed in the classical setting in the seminal paper of Lions [7].

## 2 Notations

- $N \in \mathbb{N}^*, 0 < s < 1$  and  $N \geq 2s$ .
- A constant  $C$  can vary from line to line, we will keep the same notation for it.
- The norm of  $L^p(\mathbb{R}^N)$  is denoted by  $\|\cdot\|_p$  or  $\|\cdot\|_{L^p}$
- $H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < \infty\}$  where  $\mathcal{F}$  denotes the Fourier transform, which is equivalent to

$$H^s(\mathbb{R}^N) = H^s = \{u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\}$$

endowed with the natural norm :

$$|u|_{H^s} = \left( \int |u|^2 + \int \int \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}$$

$$|\nabla_s u|_2^2 = C_{N,s} \int \int \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

$2_s^* = \frac{2N}{N-2s}$  if  $N > 2s$  and  $2_s^* = \infty$  if  $N = 2s$ .

$H^{-s}(\mathbb{R}^N) = H^{-s}$  is the dual space of  $H^s$ .

In an integral where no domain of integration is indicated,  $t$  is to be understood that the integral extends over the whole space

### 3 Proof of the main result

**Lemma 3.1** If  $F$  satisfies (F0) , then

(i) a)  $J \in C^1(H^s, \mathbb{R})$  and there exists a constant  $D > 0$  such that :

$$|J'(u)|_{H^{-s}} \leq D(|u|_{H^s} + |u|_{H^s}^{1+\frac{4s}{N}})$$

for any  $u \in H^s$ .

b)  $J^\infty \in C^1(H^s, \mathbb{R})$  and there exists a constant  $D' > 0$  such that :

$$|J'^\infty(u)|_{H^{-s}} \leq D'(|u|_{H^s} + |u|_{H^s}^{1+\frac{4s}{N}})$$

for any  $u \in H^s$ .

(ii)  $J(u) \geq A_1 |\nabla_s u|_2^2 - A_2 c^2 - A_3 c^{(1-\sigma)(\ell+2)q}$   
 $J^\infty(u) \geq B_1 |\nabla_s u|_2^2 - B_2 c^{(1-\sigma_1)(\beta+2)q_1} B_3 c^{(1-\sigma)(\ell+2)q} \quad \forall u \in S_c.$   
 $(\sigma, \sigma_1, q \text{ and } q_1 \text{ will be given below}).$

(iii) a)  $I_c > -\infty$  and any sequence of (1.1) is bounded in  $H^s$ .

b)  $I_c^\infty > -\infty$  and any minimizing sequence of (1.4) is bounded in  $H^s$ .

(iv)  $c \mapsto I_c$  and  $c \mapsto I_c^\infty$  are continuous on  $(0, \infty)$ .

**Proof :** Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by :

$$\begin{cases} \varphi(t) = 1 & \text{if } |t| < 1 \\ \varphi(t) = -|t| + 2 & \text{if } 1 \leq |t| \leq 2 \\ \varphi(t) = 0 & \text{if } |t| > 2 \end{cases}$$

$$\begin{aligned} \partial_2^1 F(x, t) &= \varphi(t) \partial_2 F(x, t) \quad \text{and} \\ |\partial_2^1 F(x, t)| &\leq A(1 + 2^{\ell+1})|t| \end{aligned} \tag{3.1}$$

$$\begin{aligned} \partial_2^2 F(x, t) &= (1 - \varphi(t)) \partial_2 F(x, t) \\ |\partial_2^2 F(x, t)| &\leq 2A|t|^{1+\frac{4s}{N}} \end{aligned} \tag{3.2}$$

Let

$$p = \begin{cases} \frac{2N}{N+2s} & \text{for } N > 2s \\ \frac{4}{3} & \text{if } N = 2s \end{cases}$$

and  $q = (1 + \frac{4s}{N})p$ .

(3.1) and (3.2) imply that  $\partial_2^1 F(x, \cdot) \in C(L^2, L^2)$  and  $\partial_2^2 F(x, \cdot) \in C(L^q, L^p)$  and there exists a constant  $K > 0$  such that :

$$|\partial_2^1 F(x, u)|_2 \leq K|u|_2 \quad \forall u \in L^2$$

$$|\partial_2^2 F(x, u)|_p \leq K|u|_q^{1+\frac{4s}{N}} \quad \forall u \in L^q.$$

Noticing that  $H^s$  is continuously embedded in  $L^q$  since  $q \in [2, \frac{2N}{N-2s}]$  for  $N > 2s$  and  $q \in [2, \infty)$  for  $N = 2s$  and  $L^p$  is continuously embedded in  $H^{-s}$  since  $p' \in [2, \frac{2N}{N-2s}]$  for  $N > 2s$  and  $p' \in [2, \infty)$  for  $N = 2s$ . We can assert that :

$$\partial_2^1 F(x, \cdot) + \partial_2^2 F(x, \cdot) \in C(H^s H^{-s}),$$

and there exists a constant  $C > 0$  such that

$$|\partial_2 F(x, u)|_{H^{-s}} \leq C\{|u|_{H^s} + |u|_{H^s}^{1+\frac{4s}{N}}\} \tag{3.3}$$

for all  $u \in H^s$ .

On the other hand :

$$\int F(x, u) \leq A(|u|_2^2 + |u|_{\ell+2}^{\ell+2}) \leq C(|u|_{H^s}^2 + |u|_{H^s}^{\ell+2})$$

, which implies that  $J \in C^1(H^s, \mathbb{R})$  by standard arguments of differential calculus.

Therefore,

$$|J'(u)|_{H^{-s}} \leq C\{|u|_{H^s} + |u|_{H^s}^{1+\frac{4s}{N}}\} \quad \forall u \in H^s.$$

- (i) b) can be easily deduced following the same steps which yield to similar estimates as (3.1) and (3.2)
- (ii) These estimates were obtained in [4].
- (iii) is a direct consequence of (ii)

**Proof of (iv)**

Consider  $c > 0$  and a sequence  $\{c_n\} \subset (0, \infty)$  such that  $c_n \rightarrow c$ . For any  $n \in \mathbb{N}$ , there exists  $u_n \in S_{c_n}$  such that  $I_{c_n} \leq J(u_n) \leq I_{c_n} + \frac{1}{n}$ . By (iii), there exists  $K > 0$  such that  $|u_n|_{H^s} \leq K$  for all  $n \in \mathbb{N}$ . Setting  $w_n = \frac{c}{c_n}u_n$ , we have that  $w_n \in S_c$  and  $|u_n - w_n|_{H^s} \leq |1 - \frac{c}{c_n}| |u_n|_{H^s} \leq K|1 - \frac{c}{c_n}|$  for any  $n \in \mathbb{N}$ .

Therefore, there exists  $n_1$  such that  $|u_n - w_n|_{H^s} \leq K + 1$  for all  $n \geq n_1$ . By part (i), there exists a constant  $L(K) > 0$  such that  $\|J'(u)\|_{H^{-s}} \leq L(K)$  for all  $u \in H^s$  such that  $|u|_{H^s} \leq 2K + 1$ .

So for all  $n \geq n_1$  :

$$\begin{aligned}
|J(w_n) - J(u_n)| &= \left| \int_0^1 \frac{d}{dt} J(tw_n + (1-t)u_n) dt \right| \\
&\leq \sup_{\|u\|_{H^s} \leq 2K+1} \|J'(u)\|_{H^{-s}} \|u_n - w_n\|_{H^s} \\
&\leq L(K)K|1 - \frac{c}{c_n}|
\end{aligned}$$

and so  $\liminf I_{c_n} \geq I_c$ . (3.4)

On the other hand there exists a sequence  $\{u_n\} \subset S_c$  such that  $J(u_n) \rightarrow I_c$  and thus by (iii), we can find  $K > 0$  such that  $|u_n|_{H^s} \leq K$ .  $w_n = \frac{c_n}{c}u_n$ . As above, we can write  $w_n \in S_{c_n}$  and  $\|u_n - w_n\|_{H^s} \leq K|1 - \frac{c_n}{c}|$

$$I_{c_n} \leq J(w_n) \leq J(u_n) + L(K)L|1 - \frac{c_n}{c}|,$$

proving that  $\limsup I_{c_n} \leq \lim J(u_n) = I_c$ . This together with (3.4) imply that

$$\lim_{n \rightarrow \infty} I_{c_n} = I_c.$$

**Lemma 3.2**

1. If  $F$  satisfies (F0) and (F1), then  $I_c < 0$  for any  $c > 0$ .
2. If  $F$  satisfies (F1) and (F4), then  $I_c^\infty < 0$  for any  $c > 0$ .

**Proof :** Let  $\varphi$  be a non-negative, radial and radially decreasing function belonging to  $S_c$ .

Let  $0 < \lambda \ll 1$  and set  $\varphi_\lambda(x) = \lambda^{N/2}\varphi(\lambda x)$  then  $\varphi_\lambda \in S_c$  and

$$\begin{aligned}
J(\varphi_\lambda) &= C_{N,s} \int \int \frac{|\lambda^{N/2}\varphi(\lambda x) - \lambda^{N/2}\varphi(\lambda y)|^2}{|x - y|^{N+2s}} dx dy \\
&\quad - \int F(x, \lambda^{N/2}\varphi(\lambda x)) dx \\
J(\varphi_\lambda) &\leq C_{N,s} \int \int \lambda^N \frac{|\varphi(\lambda x) - \varphi(\lambda y)|^2}{|x - y|^{N+2s}} dx dy \\
&\quad - \int_{|x| \geq R} F(x, \lambda^{N/2}\varphi(\lambda x)) dx \\
&\leq C_{N,s} \lambda^{2s} \int \int \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx dy \\
&\quad - \Delta \lambda^{N/2\alpha} \int_{|x| \geq R} |x|^{-p} \varphi^\alpha(\lambda x) dx \\
&\leq \lambda^{2s} |\nabla_s \varphi|_2^2 - \Delta \lambda^{\frac{N}{2}\alpha - N} \lambda^p \int_{|y| \geq \lambda R} |y|^{-p} \varphi^\alpha(y) dy
\end{aligned}$$

since  $0 < \lambda \ll 1$ , we certainly have :

$$\begin{aligned}
J(\varphi_\lambda) &\leq \lambda^{2s} |\nabla_s \varphi|_2^2 - \Delta \lambda^{\frac{N}{2}\alpha - N + p} \int_{|y| \geq R} |y|^{-p} \varphi^\alpha(y) dy \\
&\leq \lambda^{2s} \{C_1 - \lambda^{\frac{N}{2}\alpha - N + p - 2s} C_2\}
\end{aligned}$$

letting  $\lambda \rightarrow 0$  and using the fact that  $N + 2s > \frac{N}{2}\alpha + p$  the strict negativity of  $I_c$  follows.

b) The proof is dential.

**Lemma 3.3**

1. If  $F$  satisfies (F0), (F1) and (F2) then

$$I_c \leq I_a + I_{c-a} \quad \forall a \in (0, c) \quad (3.5)$$

2. If  $F$  satisfies (F2), (F4) and (F1) holds true for  $F^\infty$  then :

$$I_c^\infty < I_a^\infty + I_{c-a}^\infty \quad \forall a \in (0, c) \quad (3.6)$$



**Proof :**

1. This is a direct consequence of the fact that a real-valued function  $f$  satisfying  $f(\theta t) \leq \theta^2 f(t)$  for any  $\theta \geq 1$  does certainly verify :

$$f(c) \leq f(a) + f(a - c) \quad \forall a \in (0, c), \quad [7]$$

2. Following the same steps as in the last part, we can conclude that :

$$I_{\theta c}^\infty < \theta^2 I_c^\infty \quad \forall \theta > 1.$$

Let  $c > 0, 0 < a < c$  and  $\theta > 1$ , we can choose  $\varepsilon > 0$  such that  $\varepsilon < -I_c^\infty(1 - \theta^{-\sigma})$  and there exists  $v \in S_c$  such that :  $I_c^\infty < J^\infty(v) < I_c^\infty + \varepsilon$ .

Hence

$$I_{\theta c}^\infty \leq J^\infty(\theta v) \leq \theta^{\sigma+2} J^\infty(v).$$

Therefore  $I_{\theta c}^\infty \leq \theta^{\sigma+2} \{I_c^\infty + \varepsilon\} < \theta^{\sigma+2} I_c^\infty$  by the choice of  $\varepsilon$ .

**Proof of Theorem 1.2**

Let  $(u_n)$  be a minimizing sequence of the problem (1.4).

**Vanishing does not occur :**

If it occurs it follows from Lemma I.1 of [7] that  $|u_n|_p \rightarrow 0$  as  $n \rightarrow +\infty$  for  $p \in (2, 2_s^*)$ . By (F4)

$$\int F^\infty(x, u_n) \leq B\{|u_n|_{\gamma+2}^{\gamma+2} + |u_n|_{\ell+2}^{\ell+2}\}.$$

Thus  $\lim_{n \rightarrow +\infty} \int F^\infty(x, u_n) = 0$ , which implies that  $\liminf J^\infty(u_n) \geq 0$ , contradicting the fact that  $I_c^\infty < 0$ .

**Dichotomy does not occur :**

We will use the notation introduced in the appendix :

$$\text{For } n \geq n_0 : J^\infty(u_n) - J^\infty(v_n) - J^\infty(w_n) =$$

$$\frac{1}{2} \int |\nabla_s u_n|^2 - |\nabla_s v_n|^2 - |\nabla_s w_n|^2 - \int F^\infty(x, u_n) - F^\infty(x, v_n) - F^\infty(x, w_n)$$

$$\frac{1}{2} \int |\nabla_s u_n|^2 - |\nabla_s v_n|^2 - |\nabla_s w_n|^2 - \int F^\infty(x, u_n) - F^\infty(x, v_n + w_n)$$

since  $\text{supp } v_n \cap \text{supp } w_n = \emptyset$

$$\geq -\varepsilon - \int F^\infty(x, u_n) - F^\infty(x, v_n + w_n).$$

Now since  $\{v_n\}$  and  $\{w_n\}$  are also bounded in  $H^s$ , it follows from the proof of Lemma 3.1 that there exists  $C, K > 0$  such that :

$$\begin{aligned}
& \left| \int F^\infty(x, u_n) - F^\infty(x, v_n + w_n) \right| \\
& \leq \sup_{|u|_{H^s} \leq K} |\partial_2 F^\infty(x, u)|_{H^{-s}} |u_n - (v_n + w_n)|_{H^s} \\
& \leq \sup_{|u|_{H^s} \leq K} |\partial_2^1 F^\infty(x, u)|_{L^2} |u_n - (v_n + w_n)|_{L^2} \\
& + \sup_{|u|_{H^s} \leq K} |\partial_2^2 F^\infty(x, u)|_{L^p} |u_n - (v_n + w_n)|_{L^{p'}} \\
& \leq C \sup_{|u|_{H^s} \leq K} |u|_{L^2} |u_n - (v_n + w_n)|_{L^2} \\
& + C \sup_{|u|_{H^s} \leq K} |u|_{L^q}^{1+\frac{4s}{N}} |u_n - (v_n + w_n)|_{L^{p'}} \\
& \leq C_1 K |u_n - (v_n + w_n)|_{L^2} + C_2 K^{1+\frac{4s}{N}} |u_n - (v_n + w_n)|_{L^{p'}}
\end{aligned}$$

so :

$$J^\infty(u_n) - J^\infty(v_n) - J^\infty(w_n) \geq$$

$$-\varepsilon - C_1 K |u_n - (v_n + w_n)|_{L^2} + C_2 K^{1+\frac{4s}{N}} |u_n - (v_n + w_n)|.$$

Given any  $\delta > 0$ , we can find  $\varepsilon_\delta \in (0, \delta)$  such that (we have used the properties of the sequences  $(v_n)$  and  $(w_n)$ )

$$J^\infty(u_n) - J^\infty(v_n) - J^\infty(w_n) \geq -\delta.$$

Now let

$$a_n^2(\delta) = \int v_n^2$$

$$b_n^2(\delta) = \int w_n^2.$$

Passing to a subsequence, we may suppose that :

$$a_n^2(\delta) \rightarrow a^2(\delta)$$

and

$$b_n^2(\delta) \rightarrow b^2(\delta)$$

where

$$|a^2(\delta) - a^2| \leq \varepsilon_\delta < \delta \text{ and } |b^2(\delta) - (c^2 - a^2)| < \varepsilon$$

Recalling that  $I_c^\infty$  is continuous, we find that :

$$\begin{aligned} I_c^\infty &\geq \lim_{n \rightarrow \infty} J^\infty(u_n) \geq \liminf \{J^\infty(v_n) + J^\infty(w_n)\} \\ &\geq I_{a(\delta)}^\infty + I_{b(\delta)}^\infty - \delta. \end{aligned}$$

Letting  $\delta$  goes to zero and using again the continuity of  $I_c^\infty$ , we obtain :

$$I_c^\infty \geq I_a^\infty + I_{\sqrt{c^2 - a^2}}^\infty$$

contradicting Lemma 3.3.

Hence **compactness occurs** : so there exists  $\{y_n\} \subset \mathbb{R}^N$  such that for all  $\varepsilon > 0$  :

$$\int_{B(y_n, R(\varepsilon))} u_n^2 \geq c^2 - \varepsilon.$$

For each  $n \in \mathbb{N}$ , we can choose  $z_n \in \mathbb{Z}^N$  such that  $y_n - z_n \in [0, 1]^N$ .

Now let  $v_n = u_n(x + z_n)$ , we certainly have that  $|v_n|_{H^s} = |u_n|_{H^s}$  is bounded and so passing to a subsequence, we may assume that  $(v_n)$  converges weakly to  $v$  in  $H^s$  in particular  $(v_n)$  converges weakly to  $v$  in  $L^2$  and  $|v_n|_2^2 = c^2$ , but

$$\begin{aligned} \int v^2 &\geq \int_{B(0, R(\varepsilon) + \sqrt{N})} |v|^2 \\ &= \lim_{n \rightarrow \infty} \int_{B(0, R(\varepsilon) + \sqrt{N})} |v_n|^2 = \lim_{n \rightarrow \infty} \int_{B(z_n, R(\varepsilon) + \sqrt{N})} |v_n|^2 \end{aligned}$$

and

$$\int_{B(z_n, R(\varepsilon) + \sqrt{N})} u_n^2 \geq \int_{B(y_n, R(\varepsilon))} u_n^2 \geq c^2 - \varepsilon$$

since  $|y_n - z_n| \leq \sqrt{N}$ .

Hence  $|v|_{L^2}^2 \geq c^2 - \varepsilon \forall \varepsilon > 0 \Rightarrow |v|_{L^2}^2 \geq c^2$ .

On the other hand  $|v|_2 \leq \liminf |v_n|_2 \Rightarrow |v|_{L^2}^2 \leq c^2$ .

It follows then that  $|v|_{L^2}^2 = c^2 \Rightarrow |v - v_n|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$ .

Furthermore by the periodicity of  $F^\infty$  :

$$J^\infty(u_n) = J^\infty(v_n) \rightarrow I_c^\infty$$

and

$$v_n \rightarrow v \text{ in } L^p, p \in [2, 2_s^*).$$

It follows that  $v_n \rightarrow v$  in  $H^s$  and consequently  $\int F^\infty(x, v_n) \rightarrow \int F^\infty(x, v)$ , which implies that

$$J^\infty(v) = I_c^\infty.$$

**Lemma 3.4**

If  $F$  satisfies (F0), (F1), (F2) and (1.4) is achieved then

$$I_c < I_a + I_{c-a}^\infty \quad \forall a \in (0, c).$$

**Proof of Theorem 1.1**

In the following  $(u_n)$  is a minimizing sequence of (1.1) and we will make use of the notation introduced in the appendix.

**Vanishing does not occur :**

If it occurs, it would follow from Lemma I.1 of [7] that  $|u_n|_{L^p} \rightarrow 0$  for  $p \in (2, 2_s^*)$ .

Combining (F0) and (F3) we have : For any  $\delta > 0, \exists R_\delta > 0$  such that

$$F(x, t) \leq \delta(t^2 + |t|^{\beta+2}) + A'(|t|^{\gamma+2} + |t|^{\ell+2}) \quad \forall |x| \geq R_\delta.$$

Hence

$$\int_{|x| \geq R_\delta} F(x, u_n) \leq \delta(|u_n|_2^2 + |u_n|_{\beta+2}^{\beta+2}) + A'(|u_n|_{\beta+2}^{\beta+2} + |u_n|_{\ell+2}^{\ell+2}).$$

Thus

$$\limsup_{n \rightarrow \infty} \int_{|x| \geq R_\delta} F(x, u_n) \leq \delta c^2.$$

On the other hand :

$$\begin{aligned} \int_{|x| \leq R_\delta} F(x, u_n) &\leq A \int_{|x| \leq R_\delta} |u_n|^2 + |u_n|^{\ell+2} \\ &\leq A \left\{ |u_n|_{\ell+2}^{\ell+2} |R_\delta|^{\frac{\ell}{\ell+2}} + |u_n|_{\ell+2}^{\ell+2} \right\} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Hence for any  $\delta > 0$  we have that

$$\limsup_{n \rightarrow \infty} \int F(x, u_n) < \delta c^2$$

and so  $\lim \int F(x, u_n) = 0$ .

But  $J(u_n) \rightarrow I_c < 0$  and we obtain the contradiction.

**Dichotomy does not occur :**

Suppose that the sequence  $\{y_n\}$  is bounded and let us consider :

$$\begin{aligned}
J(u_n) - J(v_n) - J^\infty(w_n) &= \frac{1}{2} \{ |\nabla_s u_n|_2^2 - |\nabla_s v_n|_2^2 - |\nabla_s w_n|_2^2 \} \\
&- \int F(x, u_n) - F(x, v_n) - F(x, w_n) \\
&+ \int F^\infty(x, w_n) - F(x, w_n) \\
&\geq -\varepsilon - \int F(x, u_n) - F(x, v_n + w_n) + \int F^\infty(x, w_n) - F(x, w_n)
\end{aligned}$$

since  $\text{supp } v_n \cap \text{supp } w_n = \emptyset$

$$\geq -\varepsilon - \int F(x, u_n) - F(x, v_n + w_n) + \int_{|x-y_n| \geq R_n} F^\infty(x, w_n) - F(x, w_n).$$

Now using the same argument as before, it follows that :

Given  $\delta > 0$ , we can choose  $\varepsilon = \varepsilon_\delta \in (0, \delta)$  such that  $-\varepsilon - \int F(x, u_n) - F(x, v_n + w_n) \geq -\delta$  and hence  $J(u_n) - J(v_n) - J^\infty(w_n) \geq -\delta + \int_{|x-y_n| \geq R_n} F^\infty(x, w_n) - F(x, w_n)$ .

Given  $\eta > 0$ , we can find  $R > 0$  such that for all  $t \in \mathbb{R}$  and  $|x| \geq R$

$$|F^\infty(x, t) - F(x, t)| \leq \eta(t^2 + |t|^{\beta+2}).$$

Now since  $R_n \rightarrow \infty$  and we are supposing that  $\{y_n\}$  is bounded, we have that :

$$\{x : |x - y_n| \geq R_n\} \subset \{x : |x| \geq R\}$$

for  $n$  large enough.

From this and the boundedness of  $w_n$  in  $H^s$ , it follows that

$$\lim_{n \rightarrow \infty} \int_{|x-y_n| \geq R_n} F^\infty(x, w_n) - F(x, w_n) = 0.$$

Now let

$$a_n^2(\delta) = \int v_n^2$$

$$b_n^2(\delta) = \int w_n^2.$$

Passing to a subsequence, we may suppose that :

$$\begin{aligned} a_n^2(\delta) &\rightarrow a^2(\delta) \\ b_n^2(\delta) &\mapsto b^2(\delta) \end{aligned}$$

where  $|a_n^2(\delta) - a^2| < \varepsilon_\delta < \delta$  and  $|b_n^2(\delta) - (c^2 - a^2)| \leq \varepsilon_\delta < \delta$ .

Recalling that  $I_c$  and  $I_c^\infty$  are continuous, we find that :

$$\begin{aligned} I_c &= \lim_{n \rightarrow \infty} J(u_n) \geq \liminf \{J(v_n) + J^\infty(w_n)\} - \delta \\ &\geq \liminf \{I_{a_n(\delta)} + I_{b_n(\delta)}\} - \delta \end{aligned}$$

Thus  $I_c \geq I_a + I_{\sqrt{c^2 - a^2}} - \delta$ .

Letting  $\delta \rightarrow 0$  we get  $I_c \geq I_a + I_{\sqrt{c^2 - a^2}}$ .

Thus the sequence  $\{y_n\}$  cannot be bounded and, passing to a subsequence, we may suppose that  $|y_n| \rightarrow \infty$ . Now we obtain a contradiction with Lemma 3.4 by using similar arguments applied to  $J(u_n) - J^\infty(v_n) - J(w_n)$  to show that  $I_c \geq I_a^\infty + I_{\sqrt{c^2 - a^2}}$ .

Thus dichotomy cannot occur and we have **compactness**.

According to the appendix, there exists  $\{y_n\} \subset \mathbb{R}^N$  such that

$$\int_{B(y_n, R(\varepsilon))} u_n^2 \geq c^2 - \varepsilon \quad \forall \varepsilon > 0.$$

Let us first prove that the sequence  $\{y_n\}$  is bounded . If it is not the case, we may assume that  $|y_n| \rightarrow \infty$  by passing to a subsequence. Now we can choose  $z_n \in \mathbb{Z}^N$  such that  $y_n - z_n \in [0, 1]^N$ .

Setting  $v_n(x) = u_n(x + z_n)$ , we can suppose that  $(v_n)$  converges weakly to  $v$  in  $H^s$  and  $\|v_n - v\|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$  for  $2 \leq p \leq 2_s^*$ . Of course  $J^\infty(v_n) = J^\infty(u_n)$ .

On the other hand  $J(u_n) - J^\infty(u_n) = \int F^\infty(x, u_n) - F(x, u_n) = \int F^\infty(x, v_n) - F(x - z_n, v_n)$ .

Now given  $\varepsilon > 0$ , it follows from (F3) that there exists  $R > 0$  such that :

$$\begin{aligned} & \left| \int_{|x - z_n| \geq R} F^\infty(x, v_n) - F(x - z_n, v_n) \right| = \\ & \left| \int_{|x - z_n| \geq R} F^\infty(x - z_n, v_n) - F(x - z_n, v_n) \right| \leq \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon \int_{|x-z_n| \geq R} |v_n|^2 + |v_n|^{\beta+2} \leq \varepsilon C \{|v_n|_{H^s}^2 + |v_n|_{H^s}^{\beta+2}\} \\
&\leq \varepsilon D \text{ since } (v_n) \text{ is bounded in } H^s
\end{aligned}$$

On the other hand since  $|z_n| \rightarrow \infty$ , there exists  $n_R > 0$  such that for all  $n \geq n_R$ :

$$\begin{aligned}
& \left| \int_{|x-z_n| \leq R} F^\infty(x, v_n) - F(x - z_n, v_n) \right| \\
&\leq \int_{|x| \geq \frac{1}{2}|z_n|} F^\infty(x, v_n) - F(x - z_n, v_n) \\
&\leq A \int_{|x| \geq \frac{1}{2}|z_n|} |v_n|^2 + |v_n|^{\ell+2} \\
&\leq A \int_{|x| \geq \frac{1}{2}|z_n|} |v|^2 + |v|^{\ell+2} + A \int_{|x| \geq \frac{1}{2}|z_n|} |v - v_n|^2 + |v - v_n|^{\ell+2} \\
&\leq A \int_{|x| \geq \frac{1}{2}|z_n|} |v|^2 + |v|^{\ell+2} + A \int_{\mathbb{R}^N} |v - v_n|^2 + |v - v_n|^{\ell+2}
\end{aligned}$$

and hence

$$\lim \left| \int_{|x-z_n| \geq R_n} F^\infty(x, v_n) - F(x - z_n, v_n) \right| = 0.$$

Thus

$$\liminf \{J(u_n) - J^\infty(u_n)\} \geq -\varepsilon D \quad \forall \varepsilon > 0.$$

And so  $I_c = \lim J(u_n) \geq \lim J^\infty(u_n) \geq I_c^\infty$  contradicting the fact that  $I_c < I_c^\infty$ .

Hence  $\{y_n\}$  is bounded. Set  $\rho = \sup_{n \in \mathbb{N}} |y_n|$ , it follows that

$$\int_{B(0, R(\varepsilon) + \rho)} u_n^2 \geq \int_{B(y_n, R(\varepsilon))} u_n^2 \geq c^2 - \varepsilon \quad \forall \varepsilon > 0.$$

Thus

$$\begin{aligned}
\int u^2 &\geq \int_{B(0, R(\varepsilon) + \rho)} u^2 = \lim_{n \rightarrow \infty} \int_{B(0, R(\varepsilon) + \rho)} u_n^2 \\
&\geq c^2 - \varepsilon \quad \forall \varepsilon > 0.
\end{aligned}$$

and hence  $\int u^2 \geq c^2$ , on the other hand  $\int u^2 \leq c^2$ . Thus  $u \in S_c$  and  $|u_n - u|_{L^2} \rightarrow 0$ . By the boundedness of  $u_n$  in  $H^s$ , it follows that  $u_n \rightarrow u$  in  $L^p$  for  $p \in [2, 2_s^*]$ , therefore

$$\lim_{n \rightarrow \infty} \int F(x, u_n) = \int F(x, u), \text{ implying}$$

that  $J(u) = I_c$ .

## Appendix

The concentration compactness Lemma :

If  $(u_n)$  is a bounded sequence in  $H^s$  such that  $\int u_n^2 = c^2$ , then one of the following alternatives occur.

1. **Vanishing** :  $\limsup_{y \in \mathbb{R}^N} \int_{y+B_R} u_n^2 = 0$ .
2. **Dichotomy** : There exists  $a \in (0, c)$  such that  $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$  and two bounded sequences in  $H^s$  denoted by  $v_n$  and  $w_n$  (all depending on  $\varepsilon$ ) such that for every  $n \geq n_0$ , we have

$$|\int v_n^2 - a^2| < \varepsilon \text{ and } |\int w_n^2 - (c^2 - a^2)| < \varepsilon$$

$$||\nabla_s u_n|^2 - |\nabla_s v_n|^2 - |\nabla_s w_n|^2| \geq -2\varepsilon$$

and

$$|u_n - (v_n - w_n)|_p \leq 4\varepsilon \quad \forall p \in [2, 2_s^*].$$

Furthermore  $\exists (y_n) \subset \mathbb{R}^N$  and  $\{R_n\} \subset (0, \infty)$  such that  $\lim_{n \rightarrow \infty} R_n = +\infty$  and :

$$\begin{cases} v_n = u_n & \text{if } |x - y_n| \leq R_0 \\ |v_n| \leq |u_n| & \text{if } R_0 \leq |x - y_n| \leq 2R_0 \\ v_n = 0 & \text{if } |x - y_n| \geq 2R_0 \end{cases}$$

$$\begin{cases} w_n = 0 & \text{if } |x - y_n| \leq R_n \\ |w_n| \leq |u_n| & \text{if } R_n \leq |x - y_n| \leq 2R_n \\ w_n = v_n & \text{if } |x - y_n| \geq 2R_n \end{cases}$$

with  $\text{dist}(\text{supp}|v_n|, \text{supp}(w_n)) \rightarrow \infty$  as  $n \rightarrow \infty$ .



**Compactness** : There exists a sequence  $\{y_n\} \subset \mathbb{R}^N$  such that for all  $\varepsilon > 0$ , there exists  $R(\varepsilon) > 0$  such that

$$\int_{B(y_n, R(\varepsilon))} u_n^2 \geq c^2 - \varepsilon.$$

## References

1. C. J. Amick and J. F. Toland, Uniqueness and related analytic properties for the Benjamin-Ono equation-a nonlinear Neumann problem in the plane, *Acta Math.*, 167 (1991), pp. 107-126.
2. L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut, Nonlocal models for nonlinear, dispersive waves, *Phys. D*, 40 (1989), pp. 360-392.
3. Eleonora Di Nezza, G. Patalucci, E. Valdinocci, *Hitchhiker's guide To the fractional Sobolev spaces*.
4. M. Fall, personal communications.
5. H. Hajaiej, Variational problems related to some fractional kinetic equations, Preprint.
6. H. Hajaiej, L. Molinet, T. Ozawa, B. Wang, Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized Boson equations, Preprint.
7. P. L. Lions, P.L. Lions, The concentration-compactness principle in the calculus of variations, the locally compact case, Part 1( p 109-145) and Part2( p 223-281). *Ann Ins H Poincare* Vol 1 N4, 1984.